

# COMPLETE CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

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**ABSTRACT.** We exhibit 3 families of complete curvature homogeneous pseudo-Riemannian manifolds which are modeled on irreducible symmetric spaces and which are not locally homogeneous. All of the manifolds have nilpotent Jacobi operators; some of the manifolds are, in addition, Jordan Osserman and Jordan Ivanov-Petrova.

## 1. INTRODUCTION

Consider a triple  $\mathcal{U} := (V, g, A)$  where  $g$  is a non-degenerate inner product of signature  $(p, q)$  on an  $m$ -dimensional real vector space  $V$  with  $m = p + q$  and where  $A \in \otimes^4 V^*$  is an algebraic curvature tensor – i.e. a 4-tensor satisfying the usual symmetries of the Riemann curvature tensor:

$$\begin{aligned} A(x, y, z, w) &= -A(y, x, z, w) = A(z, w, x, y) \quad \text{and} \\ A(x, y, z, w) &+ A(y, z, x, w) + A(z, x, y, w) = 0. \end{aligned}$$

We also consider a pair  $\mathcal{M} := (M, g_M)$  where  $g_M$  is a pseudo-Riemannian metric of signature  $(p, q)$  on a manifold  $M$  of dimension  $m = p + q$ . One says  $\mathcal{M}$  is *Riemannian* if  $p = 0$  and *Lorentzian* if  $p = 1$ . Let  $R_M$  be the associated Riemann curvature tensor. We say that  $\mathcal{U}$  is a *0-model* for  $\mathcal{M}$  if for every point  $P \in M$ , there exists an isomorphism  $\Phi_P : T_P M \rightarrow V$  so that

$$\Phi_P^* g = g_M|_{T_P M} \quad \text{and} \quad \Phi_P^* A = R_M|_{T_P M}.$$

One says that  $\mathcal{M}$  is *curvature homogeneous* if  $\mathcal{M}$  admits a 0-model, in other words, the metric and curvature tensor “look the same at each point”. If  $\mathcal{N} := (N, g_N)$  is a homogeneous space, we say that  $\mathcal{M}$  is *modelled on*  $\mathcal{N}$  if  $(T_Q N, h_N|_{T_Q N}, R_N|_{T_Q N})$  is a 0-model for  $(M, g_M, R_M)$ ; the precise  $Q \in N$  being immaterial since  $\mathcal{N}$  is assumed to be homogeneous. We refer to [14, 16, 18] for further details.

We say that  $A^1 \in \otimes^5 V^*$  is an *algebraic covariant derivative curvature tensor* if  $A^1$  has the curvature symmetries of the covariant derivative of the Riemann curvature tensor, i.e. we have the relations:

$$\begin{aligned} A^1(x, y, z, w; v) &= A^1(z, w, x, y; v) = -A^1(y, x, z, w; v), \\ A^1(x, y, z, w; v) &+ A^1(y, z, x, w; v) + A^1(z, x, y, w; v) = 0, \\ A^1(x, y, z, w; v) &+ A^1(x, y, w, v; z) + A^1(x, y, v, z; w) = 0. \end{aligned}$$

We say that a quadruple  $\mathcal{U}^1 := (V, g, A, A^1)$  is a *1-model* for  $\mathcal{M}$  if for every point  $P \in M$ , there exists an isomorphism  $\Phi_P : T_P M \rightarrow V$  so that

$$\Phi_P^* g = g_M|_{T_P M}, \quad \Phi_P^* A = R_M|_{T_P M}, \quad \text{and} \quad \Phi_P^* A^1 = \nabla R_M|_{T_P M}.$$

In this setting,  $\mathcal{M}$  is said to be *1-curvature homogeneous*. The notion of  $k$ -curvature homogeneous for  $k \geq 2$  is defined similarly. These notions were first introduced by Singer who showed:

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**thm-1.1**

**Theorem 1.1. (Singer [15])** *There exists an universal bound  $k_m$  such that a Riemannian manifold  $\mathcal{M}$  of dimension  $m$  is locally homogeneous if and only if  $\mathcal{M}$  is  $(k_m + 1)$ -curvature homogeneous. Furthermore,  $k_m$  is smaller than  $m(m - 1)/2$ .*

One has the following important results in the context of models based on the curvature tensors of irreducible symmetric spaces in the Riemannian and Lorentzian setting:

**thm-1.2**

**Theorem 1.2.**

- (1) **(Tricerri and Vanhecke [17])** *A Riemannian curvature homogeneous manifold modelled on an irreducible symmetric space is locally symmetric.*
- (2) **(Cahen, Leroy, Parker, Tricerri, and Vanhecke [3])** *A Lorentzian curvature homogeneous manifold modelled on an irreducible symmetric space has constant sectional curvature.*

The proof of Theorem 1.2 (1) uses properties of the scalar curvature invariants of Riemannian manifolds which do not hold for indefinite metrics; the proof of Theorem 1.2 (2) uses the remark, which is based on M. Berger's classification, that any irreducible Lorentzian symmetric space of dimension greater than or equal to 3 has constant sectional curvature. In this brief note, we will present several examples illustrating that Theorem 1.2 fails in the higher signature setting by constructing complete curvature homogeneous pseudo-Riemannian manifolds which are modelled on irreducible symmetric spaces and which are not locally homogeneous.

Throughout this paper, we will be introducing metrics, curvature tensors, and covariant derivative curvature tensors. In the interests of brevity, we shall often only give the non-zero components of these tensors up to the usual  $\mathbb{Z}_2$  symmetries.

**sect-1.1**

**1.1. Signature  $(p, p)$ .** There are curvature homogeneous pseudo-Riemannian manifolds of balanced (or neutral) signature  $(p, p)$  which are complete but not locally homogeneous (and hence not locally symmetric), but which nevertheless are modelled on a complete irreducible symmetric space.

Let  $p \geq 3$ . Let  $(\vec{x}, \vec{y})$  for  $\vec{x} = (x_1, \dots, x_p)$  and  $\vec{y} = (y_1, \dots, y_p)$  be coordinates on  $\mathbb{R}^{2p}$ . Let  $f = f(\vec{x})$  be a smooth function on  $\mathbb{R}^p$ . Let  $\mathcal{M}_{1,p,f} := (\mathbb{R}^{2p}, g_{1,p,f})$  where

$$g_{1,p,f}(\partial_i^x, \partial_j^x) = \partial_i^x f \cdot \partial_j^x f \quad \text{and} \quad g_{1,p,f}(\partial_i^x, \partial_i^y) = 1,$$

the other components being zero. Let  $\mathbb{R}^{2p} = \text{Span}\{X_1, \dots, X_p, Y_1, \dots, Y_p\}$ . Let  $\mathcal{U}_{1,p} := (\mathbb{R}^{2p}, g_{1,p}, A_{1,p})$  where

$$g_{1,p}(X_i, Y_i) = 1 \quad \text{and} \quad A_{1,p}(X_i, X_j, X_k, X_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}.$$

The metric  $g_{1,p,f}$  and the inner product  $g_{1,p}$  have signature  $(p, p)$ .

Let  $H_f = (H_{f,ij})$ , where  $H_{f,ij} := (\partial_i^x \partial_j^x f)$ , be the Hessian matrix of second partial derivatives. Assume  $H_f > 0$ . Let  $H_f^{ij}$  be the inverse matrix. Let  $R_{1,p,f}$  be the curvature tensor of the metric  $g_{1,p,f}$  and let  $\nabla R_{1,p,f}$  be the associated covariant derivative. Set

$$\alpha_1 := \sum_{a,b,c,d,e,s,t,u,v,w} H_f^{as} H_f^{bt} H_f^{cu} H_f^{dv} H_f^{ew} \nabla R_{1,p,f}(\partial_a^x, \partial_b^x, \partial_c^x, \partial_d^x; \partial_e^x) \cdot \nabla R_{1,p,f}(\partial_s^x, \partial_t^x, \partial_u^x, \partial_v^x; \partial_w^x).$$

**thm-1.3**

**Theorem 1.3.** *Let  $p \geq 2$  and let  $H_f > 0$ . Then:*

- (1) *All geodesics in  $\mathcal{M}_{1,p,f}$  extend for infinite time.*
- (2) *If  $P \in \mathbb{R}^{2p}$ , then  $\exp_P : T_P \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$  is a diffeomorphism.*
- (3) *The non-zero components of  $R_{1,p,f}$  and of  $\nabla R_{1,p,f}$  are given by:*
  - (a)  $R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = H_{f,il} H_{f,jk} - H_{f,ik} H_{f,jl}$ ,
  - (b)  $\nabla R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x; \partial_n^x) = \partial_n^x R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x)$ .
- (4)  $\mathcal{U}_{1,p}$  *is an irreducible 0-model for  $\mathcal{M}_{1,p,f}$ .*
- (5) *If  $f = x_1^2 + \dots + x_p^2$ , then  $\mathcal{M}_{1,p,f}$  is an irreducible symmetric space.*

(6) If  $p \geq 3$  and if  $\alpha_1$  is not constant,  $\mathcal{M}_{1,p,f}$  is not curvature 1-homogeneous.

The pseudo-Riemannian manifold  $\mathcal{M}_{1,p,f}$  can be realized as a hypersurface in a flat space of signature  $(p, p+1)$ ; the Hessian  $H_f$  then gives the second fundamental form. We refer to [4, 9, 10] for further details concerning this family of manifolds.

sect-1.2

**1.2. Signature  $(2s, s)$ .** For  $s \geq 2$ , let  $(\vec{u}, \vec{t}, \vec{v})$  give coordinates on  $\mathbb{R}^{3s}$  where we have  $\vec{u} := (u_1, \dots, u_s)$ ,  $\vec{t} := (t_1, \dots, t_s)$ , and  $\vec{v} := (v_1, \dots, v_s)$ . Let  $f_i \in C^\infty(\mathbb{R})$  be smooth functions. Set

$$F(\vec{u}) := f_1(u_1) + \dots + f_s(u_s) \in C^\infty(\mathbb{R}^s) \quad \text{and} \quad |u|^2 = u_1^2 + \dots + u_s^2.$$

Let  $\mathcal{M}_{2,s,F} := (\mathbb{R}^{3s}, g_{2,s,F})$  where

$$\begin{aligned} g_{2,s,F}(\partial_i^u, \partial_j^u) &= -2\{F(\vec{u}) + \sum_{1 \leq k \leq s} u_k t_k\} \delta_{ij}, \\ g_{2,s,F}(\partial_i^u, \partial_j^v) &:= \delta_{ij}, \quad \text{and} \quad g_{2,s,F}(\partial_i^t, \partial_j^t) := -\delta_{ij}. \end{aligned}$$

Manifolds of this type were first introduced in [11]; see also [12, 13].

Let  $\mathbb{R}^{3s} = \text{Span}\{U_1, \dots, U_s, T_1, \dots, T_s, V_1, \dots, V_s\}$ . Let  $\mathcal{U}_{2,s} := (\mathbb{R}^{3s}, g_{2,s}, A_{2,s})$  for

eqn-1.a

$$(1.a) \quad \begin{aligned} g_{2,s}(U_i, V_j) &:= \delta_{ij}, \quad g_{2,s}(T_i, T_j) := -\delta_{ij}, \quad \text{and} \\ A_{2,s}(U_i, U_j, U_k, T_l) &:= \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}. \end{aligned}$$

The metric  $g_{2,s,F}$  and the inner product  $g_{2,s}$  have signature  $(2s, s)$ . Set

$$\alpha_2 := \sum_{i,j,k,l,n} \{\nabla R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_k^u, \partial_l^u, \partial_n^u)\}^2.$$

thm-1.4

**Theorem 1.4.** *Let  $s \geq 2$ . Then:*

- (1) *All geodesics in  $\mathcal{M}_{2,s,F}$  extend for infinite time.*
- (2) *If  $P \in \mathbb{R}^{3s}$ , then  $\exp_P : T_P \mathbb{R}^{3s} \rightarrow \mathbb{R}^{3s}$  is a diffeomorphism.*
- (3) *The non-zero components of  $R_{2,s,F}$  and of  $\nabla R_{2,s,F}$  are given by:*
  - (a)  $R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) = (\partial_i^u)^2 f_i + (\partial_j^u)^2 f_j + |u|^2$ ,
  - (b)  $R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^t) = 1$ ,
  - (c)  $\nabla R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u; \partial_i^u) = (\partial_i^u)^3 f_i + 4u_i$ .
- (4)  $\mathcal{U}_{2,s}$  *is an irreducible 0-model for  $\mathcal{M}_{2,s,F}$ .*
- (5) *If  $F = -\frac{1}{6}u_1^4 - \dots - \frac{1}{6}u_s^4$ , then  $\mathcal{M}_{2,s,F}$  is an irreducible symmetric space.*
- (6) *If  $s \geq 3$  and if  $\alpha_2$  is not constant,  $\mathcal{M}_{2,s,F}$  is not curvature 1-homogeneous.*

Assertion (6) in Theorem 1.4 was discussed previously in [12]; C. Dunn pointed out that the argument given there contained a mistake. In this paper, we shall give a slightly different argument which avoids that mistake.

sect-1.3

**1.3. Manifolds which are 1-curvature homogeneous.** The previous two families of examples were curvature homogeneous but not 1-curvature homogeneous for generic members of the families. Let  $r \geq 2$ . Introduce coordinates  $(\vec{u}, \vec{v}, x, y)$  on  $\mathbb{R}^{2r+2}$  where we have  $\vec{u} = (u_1, \dots, u_r)$  and  $\vec{v} = (v_1, \dots, v_r)$ . Let  $\psi \in C^\infty(\mathbb{R})$ . Let  $\mathcal{M}_{3,r,\psi} := (\mathbb{R}^{2r+2}, g_{3,r,\psi})$  where

$$\begin{aligned} g_{3,r,\psi}(\partial_x, \partial_y) &= 1, \quad g_{3,r,\psi}(\partial_{u_i}, \partial_{v_j}) = \delta_{ij}, \quad \text{and} \\ g_{3,r,\psi}(\partial_x, \partial_x) &= -2u_1 v_2 - \dots - 2u_{r-1} v_r - 2\psi(u_r). \end{aligned}$$

These manifolds are closely related to examples of Fiedler et al [5].

Let  $\mathbb{R}^{2r+2} = \text{Span}\{U_1, \dots, U_r, V_1, \dots, V_r, X, Y\}$ . Let  $\mathcal{U}_{3,r} := (\mathbb{R}^{2r+2}, g_{3,r}, A_{3,r})$  for

$$\begin{aligned} g_{3,r}(X, Y) &= 1, \quad g_{3,r}(U_i, V_j) = \delta_{ij}, \quad A_{3,r}(X, U_r, U_r, X) = 1, \quad \text{and} \\ A_{3,r}(X, U_i, V_{i+1}, X) &= 1 \quad \text{for } 1 \leq i \leq r-1. \end{aligned}$$

The metric  $g_{3,r,\psi}$  and the inner product  $g_{3,r}$  have signature  $(r+1, r+1)$ . We also define a 1-model space  $\mathcal{U}_{3,r}^1 := (\mathbb{R}^{2r+2}, g_{3,r}, A_{3,r}, A_{3,r}^1)$  where

$$A_{3,r}^1(X, U_r, U_r, X; U_r) = 1.$$

thm-1.5

**Theorem 1.5.** *Let  $r \geq 2$ . Assume that  $\psi'' > 0$ . Then:*

- (1) All geodesics in  $\mathcal{M}_{3,r,\psi}$  extend for infinite time.
- (2)  $\exp_P : T_P \mathbb{R}^{2r+2} \rightarrow \mathbb{R}^{2r+2}$  is a diffeomorphism for all  $P$  in  $\mathbb{R}^{2r+2}$ .
- (3) The non-zero components of  $R_{3,r,\psi}$  and of  $\nabla R_{3,r,\psi}$  are given by:
  - (a)  $R_{3,r,\psi}(\partial_x, \partial_{u_r}, \partial_{u_r}, \partial_x) = \psi''(u_r)$ ,
  - (b)  $R_{3,r,\psi}(\partial_x, \partial_{u_i}, \partial_{v_{i+1}}, \partial_x) = 1$  for  $1 \leq i \leq r-1$ ,
  - (c)  $\nabla R_{3,r,\psi}(\partial_x, \partial_{u_r}, \partial_{u_r}, \partial_x; \partial_{u_r}) = \psi'''(u_r)$ .
- (4)  $\mathcal{U}_{3,r}$  is an irreducible 0-model for  $\mathcal{M}_{3,r,\psi}$ .
- (5) If  $\psi(u_r) = u_r^2$ , then  $\mathcal{M}_{3,r,\psi}$  is an irreducible symmetric space.
- (6) If  $\psi''' > 0$ , then:
  - (a)  $\mathcal{U}_{3,r}^1$  is a 1-model for  $\mathcal{M}_{3,r,\psi}$ ,
  - (b)  $\mathcal{M}_{3,r,\psi}$  is not 2-curvature homogeneous.

Theorem 1.3 (6) (resp. Theorem 1.4 (6)) requires that  $p \geq 3$  (resp.  $s \geq 3$ ). This result is sharp; for suitably chosen  $f$  (resp.  $F$ ),  $\mathcal{M}_{1,2,f}$  (resp.  $\mathcal{M}_{2,2,F}$ ) is curvature 1-homogeneous but not curvature 2-homogeneous; we omit details in the interests of brevity.

**sect-1.4**

**1.4. Osserman manifolds.** If  $x$  is a tangent vector at a point  $P \in M$ , then the Jacobi operator  $J_M(x)$  is characterized by the identity

$$g_M(J_M(x)y, z) = R_M(y, x, x, z).$$

If  $\rho_M$  is the associated Ricci tensor, then  $\rho_M(x, x) = \text{Tr}(J_M(x))$ . One says that  $\mathcal{M}$  is *spacelike* (resp. *timelike*) *Osserman* if the eigenvalues of the Jacobi operator are constant on the pseudo-sphere bundle  $S^+(\mathcal{M})$  of spacelike (resp.  $S^-(\mathcal{M})$  of timelike) unit vectors. One says that  $\mathcal{M}$  is *spacelike* (resp. *timelike*) *Jordan Osserman* if the Jordan normal form of the Jacobi operator is constant on  $S^+(\mathcal{M})$  (resp.  $S^-(\mathcal{M})$ ). We shall say that  $\mathcal{M}$  is *Osserman nilpotent of order  $n$*  if  $J_M(x)^n = 0$  for every  $x \in TM$  and if there exists a point  $P_0 \in M$  and a tangent vector  $x_0 \in T_{P_0}M$  so that  $J_M(x_0)^{n-1} \neq 0$ . Such manifolds are necessarily Osserman since 0 is the only eigenvalue of  $J_M$ . And consequently such manifolds are Ricci flat since  $\rho(x, x) = \text{Tr}(J(x))$ . We refer to [6, 8] for further details concerning Osserman manifolds.

**thm-1.6**

**Theorem 1.6.**

- (1) Let  $p \geq 2$ . If  $H_f > 0$ , then  $\mathcal{M}_{1,p,f}$  is spacelike and timelike Jordan Osserman.
- (2) Let  $s \geq 2$ . Then  $\mathcal{M}_{2,s,F}$  is spacelike Jordan Osserman. However  $\mathcal{M}_{2,s,F}$  is not timelike Jordan Osserman.
- (3) Let  $r \geq 2$ . If  $\psi'' > 0$ , then  $\mathcal{M}_{3,r,\psi}$  is  $2r$ -Osserman nilpotent.

The three families  $\mathcal{M}_{i,k}$  first arose in the study of Osserman manifolds. We refer to [1, 2, 6, 7] for other examples of non-homogeneous Osserman manifolds.

**1.5. Ivanov-Petrova manifolds.** Let  $\{e_1, e_2\}$  be an oriented orthonormal basis for an oriented spacelike (resp. timelike) 2-plane  $\pi$ . The *skew-symmetric curvature operator*  $\mathcal{R}_M(\pi)$  is characterized by the identity

$$g_M(\mathcal{R}_M(\pi)y, z) = R_M(e_1, e_2, y, z).$$

This operator is independent of the particular oriented orthonormal basis chosen for  $\pi$ . One says that  $\mathcal{M}$  is *spacelike* (resp. *timelike*) *Jordan Ivanov-Petrova* if the Jordan normal form of  $\mathcal{R}_M$  is constant on the Grassmannian of oriented spacelike (resp. timelike) 2-planes; one lets the *Rank* be the common value of  $\text{rank}(\mathcal{R}_M(\pi))$  in this setting.

**thm-1.7**

**Theorem 1.7.**

- (1) Let  $p \geq 2$ . If  $H_f > 0$ , then  $\mathcal{M}_{1,p,f}$  is spacelike and timelike Jordan Ivanov-Petrova of rank 2.

- (2) Let  $s \geq 2$ . Then  $\mathcal{M}_{2,s,F}$  is spacelike Jordan Ivanov-Petrova of rank 4;  $\mathcal{M}_{2,s,F}$  is not timelike Jordan Ivanov-Petrova.

sect-1.6

**1.6. The geodesic involution.** The following observation is a special case which follows from work of E. Cartan; we present it for the sake of completeness in light of the examples given above.

thm-1.8

**Theorem 1.8.** *Let  $\mathcal{M}$  be a pseudo-Riemannian manifold of signature  $(p, q)$ . Suppose that  $\nabla R_M = 0$  and that  $\exp_P : T_P M \rightarrow M$  is a diffeomorphism for every  $P \in M$ . Then the geodesic symmetry  $\mathcal{S}_P : Q \rightarrow \exp_P\{-\exp_P^{-1}Q\}$  is an isometry. Furthermore,  $\mathcal{M}$  is a homogeneous space.*

Here is a brief guide to this paper. In Section 2, we prove Assertions (1)-(3) of Theorems 1.3-1.5. In Section 3, we show  $\mathcal{U}_{i,k}$  is a 0-model for  $\mathcal{M}_{i,k}$  and in Section 4, we show these models are irreducible. This establishes Assertion (4) of Theorems 1.3-1.5. Assertion (5) of these three Theorems then follows as a scholium to the previous assertions. In Section 5, we establish Assertion (6) of Theorems 1.3-1.5. We refer to [10] for the proof of Assertion (1) of Theorems 1.6 and 1.7 and to [12, 13] for the proof of Assertion (2) of Theorems 1.6 and 1.7. Assertion (3) of Theorem 1.6 is proved in Section 4. In Section 6, we complete our discussion by proving Theorem 1.8.

It is a pleasant task to thank Professors E. García-Río, O. Kowalski, and L. Vanhecke for useful conversations on this subject.

sect-2

## 2. COMPLETE MANIFOLDS

We shall need the following technical fact.

lem-2.1

**Lemma 2.1.** *Let  $(z_1, \dots, z_n)$  be coordinates on  $\mathbb{R}^n$ . Let  $g$  be a pseudo-Riemannian metric on  $\mathbb{R}^n$  so that  $\nabla_{\partial_a}^z \partial_b^z = \sum_{a,b < c} \Gamma_{ab}^c(z_1, \dots, z_{c-1}) \partial_c^z$ . Then:*

- (1)  $(\mathbb{R}^n, g)$  is a complete pseudo-Riemannian manifold.
- (2)  $\exp_P : T_P \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a diffeomorphism for all  $P$  in  $\mathbb{R}^n$ .

*Proof.* Let  $\gamma(t) = (z_1(t), \dots, z_n(t))$  be a curve in  $\mathbb{R}^n$ ;  $\gamma$  is a geodesic if and only

$$\begin{aligned} \ddot{z}_1(t) &= 0, \quad \text{and for } c > 1 \\ \ddot{z}_c(t) + \sum_{a,b < c} \dot{z}_a(t) \dot{z}_b(t) \Gamma_{ab}^c(z_1, \dots, z_{c-1})(t) &= 0. \end{aligned}$$

We solve this system of equations recursively. Let  $\gamma(t; \vec{z}^0, \vec{z}^1)$  be defined by

$$\begin{aligned} z_1(t) &= z_1^0 + z_1^1 t, \quad \text{and for } c > 1 \\ z_c(t) &= z_c^0 + z_c^1 t - \int_0^t \int_0^s \sum_{a,b < c} \dot{z}_a(r) \dot{z}_b(r) \Gamma_{ab}^c(z_1, \dots, z_{c-1})(r) dr ds. \end{aligned}$$

Then  $\gamma(0; \vec{z}^0, \vec{z}^1)(0) = \vec{z}^0$  while  $\dot{\gamma}(0; \vec{z}^0, \vec{z}^1)(0) = \vec{z}^1$ . Thus every geodesic arises in this way so all geodesics extend for infinite time. Furthermore, given  $P, Q \in \mathbb{R}^n$ , there is a unique geodesic  $\gamma = \gamma_{P,Q}$  so that  $\gamma(0) = P$  and  $\gamma(1) = Q$  where

$$\begin{aligned} z_1^0 &= P_1, \quad z_1^1 = Q_1 - P_1, \quad \text{and for } c > 1 \\ z_c^0 &= P_c, \quad z_c^1 = Q_c - P_c + \int_0^1 \int_0^s \sum_{a,b < c} \dot{z}_a(r) \dot{z}_b(r) \Gamma_{ab}^c(z_1, \dots, z_{c-1})(r) dr ds. \end{aligned}$$

This shows that  $\exp_P$  is a diffeomorphism from  $T_P \mathbb{R}^n$  to  $\mathbb{R}^n$ .  $\square$

*Proof of Theorem 1.3 (1-3).* Adopt the notation of Section 1.1. Let

$$g_{ij}(x) = g_{1,p,f}(\partial_i^x, \partial_j^x) := \partial_i^x f \cdot \partial_j^x f, \quad \text{and} \quad \Gamma_{ijk}(\vec{x}) := \frac{1}{2} \{ \partial_i^x g_{jk} + \partial_j^x g_{ik} - \partial_k^x g_{ij} \}.$$

The non-zero Christoffel symbols are

eqn-2.a

$$(2.a) \quad g_{1,p,f}(\nabla_{\partial_i^x} \partial_j^x, \partial_k^x) = \Gamma_{ijk}(\vec{x}) \quad \text{and} \quad \nabla_{\partial_i^x} \partial_j^x = \sum_{1 \leq k \leq p} \Gamma_{ijk}(\vec{x}) \partial_k^y.$$

We verify that the hypothesis of Lemma 2.1 is satisfied and thereby prove Assertions (1) and (2) by setting:

$$z_1 = x_1, \dots, z_p = x_p, z_{p+1} = y_1, \dots, z_{2p} = y_p.$$

Furthermore, by Equation (2.a),

$$\begin{aligned} R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) &= -\frac{1}{2}(\partial_j^x \partial_k^x g_{il} + \partial_i^x \partial_l^x g_{jk} - \partial_j^x \partial_l^x g_{ik} - \partial_i^x \partial_k^x g_{jl}) \\ &= H_{f,il} H_{f,jk} - H_{f,ik} H_{f,jl} \end{aligned}$$

while  $R_{1,p,f}(\cdot, \cdot, \cdot, \cdot) = 0$  if any of the entries is  $\partial_i^y$ . Assertion (3a) now follows. Furthermore, by Equation (2.a),

$$\nabla R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x; \partial_n^x) = \partial_n^x R_{1,p,f}(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x)$$

while  $\nabla R_{1,p,f}(\cdot, \cdot, \cdot, \cdot; \cdot) = 0$  if any of the entries is  $\partial_i^y$ . This proves Assertion (3b) of Theorem 1.3.  $\square$

*Proof of Theorem 1.4 (1)-(3).* Adopt the notation of Section 1.2. Let  $i \neq j$  and let  $g = g_{2,s,F}$ . The non-zero Christoffel symbols of the second kind are given by:

$$\begin{aligned} g(\nabla_{\partial_i^u} \partial_i^u, \partial_i^u) &= -\partial_i^u f_i - t_i, \\ g(\nabla_{\partial_i^u} \partial_j^u, \partial_j^u) &= \partial_j^u f_j + t_j, & g(\nabla_{\partial_i^u} \partial_j^u, \partial_i^u) &= g(\nabla_{\partial_j^u} \partial_i^u, \partial_i^u) = -\partial_j^u f_j - t_j, \\ g(\nabla_{\partial_i^u} \partial_i^t, \partial_i^t) &= u_i, & g(\nabla_{\partial_i^u} \partial_i^t, \partial_i^u) &= g(\nabla_{\partial_i^t} \partial_i^u, \partial_i^u) = -u_i, \\ g(\nabla_{\partial_i^u} \partial_j^t, \partial_j^t) &= u_j, & g(\nabla_{\partial_i^u} \partial_j^t, \partial_i^u) &= g(\nabla_{\partial_j^t} \partial_i^u, \partial_i^u) = -u_j. \end{aligned}$$

We may then raise indices to see the non-zero covariant derivatives are given by:

$$\begin{aligned} \nabla_{\partial_i^u} \partial_i^u &= -(\partial_i^u f_i + t_i) \partial_i^v + \sum_{k \neq i, 1 \leq k \leq s} (\partial_k^u f_k + t_k) \partial_k^v - \sum_{1 \leq k \leq s} u_k \partial_k^t, \\ \nabla_{\partial_i^u} \partial_j^u &= -(\partial_j^u f_j + t_j) \partial_i^v - (\partial_i^u f_i + t_i) \partial_j^v, \\ \nabla_{\partial_i^u} \partial_i^t &= \nabla_{\partial_i^t} \partial_i^u = -u_i \partial_i^v, \quad \text{and} \quad \nabla_{\partial_i^u} \partial_j^t = \nabla_{\partial_j^t} \partial_i^u = -u_j \partial_i^v. \end{aligned}$$

We derive Assertions (1) and (2) from Lemma 2.1 by setting:

$$z_1 = u_1, \dots, z_s = u_s, z_{s+1} = t_1, \dots, z_{2s} = t_p, z_{2s+1} = v_1, \dots, z_{3s} = v_s.$$

We have  $\nabla \partial_i^v = 0$ . Thus if at least one  $z_\mu \in \{\partial_i^v\}$ , then  $R_{2,s,F}(z_1, z_2, z_3, z_4) = 0$ . Similarly, if at least two of the  $z_\mu$  belong to  $\{\partial_i^t\}$ , then  $R_{2,s,F}(z_1, z_2, z_3, z_4) = 0$ . Finally, as  $\partial_i^u \partial_j^u F = 0$  for  $i \neq j$ ,  $R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_k^u, \star) = 0$  if the indices  $\{i, j, k\}$  are distinct. Furthermore

$$\nabla_{\partial_i^u} \nabla_{\partial_j^u} \partial_j^u = f_i'' \partial_i^v - \partial_i^t + \{|u|^2\} \partial_i^v \quad \text{and} \quad \nabla_{\partial_j^u} \nabla_{\partial_i^u} \partial_j^u = -f_j'' \partial_i^v.$$

Assertions (3a) and (3b) now follow.

We have similarly that  $\nabla R_{2,s,F}(\xi_1, \xi_2, \xi_3, \xi_4; \xi_5) = 0$  if at least one of the  $\xi_i$  belongs to  $\text{Span}\{T_i, V_i\}$ . Furthermore, the only non-zero component of  $\nabla R_{2,s,F}$  is given by:

$$\begin{aligned} &\nabla R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u; \partial_i^u) \\ &= \partial_i^u R_{2,s,F}(\partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) - 2R_{2,s,F}(\nabla_{\partial_i^u} \partial_i^u, \partial_j^u, \partial_j^u, \partial_i^u) \\ &\quad - 2R_{2,s,F}(\partial_i^u, \nabla_{\partial_i^u} \partial_j^u, \partial_j^u, \partial_i^u) \\ &= f_i''' + 2u_i + 2R_{2,s,F}(\sum_{1 \leq k \leq s} u_k \partial_k^t, \partial_j^u, \partial_j^u, \partial_i^u) + 0 = f_i''' + 4u_i. \end{aligned}$$

Assertion (3c) now follows.  $\square$

*Proof of Theorem 1.5 (1)-(3).* Adopt the notation of Section 1.3. Let  $1 \leq i \leq r-1$  and let  $g = g_{3,r,\psi}$ . We compute that the non-zero Christoffel symbols of the second kind are

$$\begin{aligned} g(\nabla_{\partial_x} \partial_x, \partial_{u_r}) &= \psi'(u_r), & g(\nabla_{\partial_x} \partial_{u_r}, \partial_x) &= g(\nabla_{\partial_{u_r}} \partial_x, \partial_x) = -\psi'(u_r), \\ g(\nabla_{\partial_x} \partial_x, \partial_{u_i}) &= v_{i+1}, & g(\nabla_{\partial_x} \partial_{u_i}, \partial_x) &= g(\nabla_{\partial_{u_i}} \partial_x, \partial_x) = -v_{i+1}, \\ g(\nabla_{\partial_x} \partial_x, \partial_{v_{i+1}}) &= u_i, & g(\nabla_{\partial_x} \partial_{v_{i+1}}, \partial_x) &= g(\nabla_{\partial_{v_{i+1}}} \partial_x, \partial_x) = -u_i. \end{aligned}$$

Consequently the non-zero Christoffel symbols of the first kind are

$$\begin{aligned} \nabla_{\partial_x} \partial_x &= u_1 \partial_{u_2} + \dots + u_{r-1} \partial_{u_r} + v_2 \partial_{v_1} + \dots + v_r \partial_{v_{r-1}} + \psi'(u_r) \partial_{v_r}, \\ \nabla_{\partial_x} \partial_{u_i} &= \nabla_{\partial_{u_i}} \partial_x = -v_{i+1} \partial_y, \quad \text{and} \quad \nabla_{\partial_x} \partial_{v_{i+1}} = \nabla_{\partial_{v_{i+1}}} \partial_x = -u_i \partial_y. \end{aligned} \quad \text{eqn-2.b}$$

To apply Lemma 2.1, we set

$$z_0 = x, \quad z_1 = u_1, \quad \dots, \quad z_r = u_r, \quad z_{r+1} = v_r, \quad \dots, \quad z_{2r} = v_1, \quad z_{2r+1} = y.$$

Assertions (1) and (2) follow. We have

$$\begin{aligned} \nabla_{\partial_{u_r}} \nabla_{\partial_x} \partial_x &= \psi'' \partial_{v_r}, & \nabla_{\partial_x} \nabla_{\partial_{u_r}} \partial_x &= 0, \\ \nabla_{\partial_{u_i}} \nabla_{\partial_x} \partial_x &= \partial_{u_{i+1}}, & \nabla_{\partial_x} \nabla_{\partial_{u_i}} \partial_x &= 0, \\ \nabla_{\partial_{v_{i+1}}} \nabla_{\partial_x} \partial_x &= \partial_{v_i}, & \nabla_{\partial_x} \nabla_{\partial_{v_{i+1}}} \partial_x &= 0. \end{aligned}$$

Assertions (3a) and (3b) follow. Assertion (3c) follows from these calculations and from Equation (2.b).  $\square$

### sect-3

### 3. MODEL SPACES

Throughout this section, we shall only list (possibly) non-zero entries of  $g$ ,  $R_g$ , and of  $\nabla R_g$  up to the usual  $\mathbb{Z}_2$  symmetries. We show  $\mathcal{U}$  is a 0-model for  $\mathcal{M}$  by exhibiting a basis for  $T_P M$  with the required normalizations for any  $P \in M$ .

#### sect-3.1

**3.1. A 0-model for  $\mathcal{M}_{1,p,f}$ .** Choose a basis  $\{X_1, \dots, X_p\}$  for  $\text{Span}\{\partial_1^x, \dots, \partial_p^x\}$  in  $T_P M$  so that  $H_f(X_i, X_j) = \delta_{ij}$ . Expand  $X_i = \sum_{1 \leq j \leq p} \xi_{ij} \partial_j^x$  and let  $\xi^{ij}$  be the inverse matrix. Set  $Y_i := \sum_{1 \leq j \leq p} \xi^{ji} \partial_j^y$ . Then

$$\begin{aligned} g_{1,p,f}(X_i, X_j) &= c_{ij}, \quad g_{1,p,f}(X_i, Y_j) = \delta_{ij}, \quad \text{and} \\ R_{1,p,f}(X_i, X_j, X_k, X_l) &= \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}, \end{aligned}$$

where  $c_{ij} = c_{ji}$ . Set  $\bar{X}_i = X_i - \frac{1}{2} \sum_{1 \leq j \leq p} c_{ij} Y_j$  and  $\bar{Y}_i := Y_i$ . We may then conclude  $\mathcal{U}_{1,p}$  is a 0-model for  $\mathcal{M}_{1,p,f}$  since

$$\begin{aligned} g_{1,p,f}(\bar{X}_i, \bar{X}_j) &= 0, \quad g_{1,p,f}(\bar{X}_i, \bar{Y}_j) = \delta_{ij}, \quad \text{and} \\ R_{1,p,f}(\bar{X}_i, \bar{X}_j, \bar{X}_k, \bar{X}_l) &= \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}. \end{aligned}$$

#### sect-3.2

**3.2. A 0-model for  $\mathcal{M}_{2,s,F}$ .** Fix  $P \in \mathbb{R}^{3s}$ . Define a new basis for  $T_P M$  by setting:

$$\text{eqn-3.a} \quad (3.a) \quad U_i := \partial_i^u + \varepsilon_i \partial_i^t + \varrho_i \partial_i^v, \quad T_i := \partial_i^t + \varepsilon_i \partial_i^v, \quad \text{and} \quad V_i := \partial_i^v$$

where the constants  $\varepsilon_i$  and  $\varrho_i$  will be specified below. Let  $i \neq j$ . Then:

$$\begin{aligned} g_{2,s,F}(U_i, T_i) &= \varepsilon_i - \varepsilon_i = 0, \quad g_{2,s,F}(U_i, U_i) = g_{2,s,F}(\partial_i^u, \partial_i^u) - \varepsilon_i^2 + 2\varrho_i, \\ g_{2,s,F}(T_i, T_i) &= -1, \quad g_{2,s,F}(U_i, V_i) = 1, \quad R_{2,s,F}(U_i, U_j, U_j, T_i) = 1, \quad \text{and} \\ R_{2,s,F}(U_i, U_j, U_j, U_i) &= (\partial_i^u)^2 f_i + (\partial_j^u)^2 f_j + |u|^2 + 2\varepsilon_i + 2\varepsilon_j. \end{aligned}$$

We set

$$\varepsilon_i := -\frac{1}{2}(\partial_i^u)^2 f_i - \frac{1}{4}|u|^2 \quad \text{and} \quad \varrho_i := \frac{1}{2}\{\varepsilon_i^2 - g_{2,s,F}(\partial_i^u, \partial_i^u)\}.$$

As  $g_{2,s,F}(U_i, U_i) = R_{2,s,F}(U_i, U_j, U_j, U_i) = 0$ ,  $\mathcal{U}_{2,s}$  is a 0-model for  $\mathcal{M}_{2,s,F}$ .

#### sect-3.3

**3.3. A 0-model for  $\mathcal{M}_{3,r,\psi}$ .** Let  $\varepsilon_i$  be real parameters to be specified below. Define a new basis  $\{X, Y, U_1, \dots, U_r, V_1, \dots, V_r\}$  for  $T_P \mathbb{R}^{2r+2}$  by setting:

$$X = \varepsilon_0 \{\partial_x - \frac{1}{2} g_{3,r,\psi}(\partial_x, \partial_x) \partial_y\}, \quad Y = \varepsilon_0^{-1} \partial_y, \quad U_i = \varepsilon_i \partial_{u_i}, \quad V_i = \varepsilon_i^{-1} \partial_{v_i}.$$

The non-zero entries in  $g_{3,r,\psi}$  are given by  $g_{3,r,\psi}(X, Y) = 1$  and  $g_{3,r,\psi}(U_i, V_i) = 1$ . We apply Theorem 1.5 (3) to see the non-zero entries in  $R_{3,r,\psi}$  and  $\nabla R_{3,r,\psi}$  are

$$\begin{aligned} R_{3,r,\psi}(X, U_r, U_r, X) &= \varepsilon_0^2 \varepsilon_r^2 \psi''(u_r), \\ R_{3,r,\psi}(X, U_i, V_{i+1}, X) &= \varepsilon_0^2 \varepsilon_i \varepsilon_{i+1}^{-1} \quad \text{for } 1 \leq i \leq r-1, \\ \nabla R_{3,r,\psi}(X, U_r, U_r, X; U_r) &= \varepsilon_0^2 \varepsilon_r^3 \psi'''(u_r). \end{aligned}$$

Assume  $\psi'' > 0$ . We can show that  $\mathcal{U}_{3,r}$  is a 0-model for  $\mathcal{M}_{3,r,\psi}$  by setting:

$$\varepsilon_r = (\psi'')^{-1/2}, \quad \varepsilon_0 = 1, \quad \text{and} \quad \varepsilon_i = \varepsilon_r \quad \text{for} \quad 1 \leq i < r.$$

If in addition we suppose that  $\psi''' \neq 0$ , then more is true. We show  $\mathcal{U}_{3,r}^1$  is a 1-model for  $\mathcal{M}_{3,r,\psi}$  by setting:

$$\varepsilon_r = \psi''(\psi''')^{-1}, \quad \varepsilon_0 = (\varepsilon_r^2 \psi'')^{-1/2}, \quad \text{and} \quad \varepsilon_i = \varepsilon_0^{-2} \varepsilon_{i+1} \quad \text{for} \quad 1 \leq i < r.$$

#### 4. IRREDUCIBILITY

sect-4

**4.1. The model  $\mathcal{M}_{1,p,f}$ .** We adopt the notation of Section 1.2. Let  $B_{1,p}$  be the algebraic curvature tensor on  $\mathbb{R}^p = \text{Span}\{X_1, \dots, X_p\}$  defined by

$$B_{1,p}(X_i, X_j, X_k, X_l) = \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}.$$

lem-4.1

**Lemma 4.1.**

- (1) Let  $0 \neq \xi_1 \in \mathbb{R}^p$ . If  $B_{1,p}(\xi_1, \xi_2, \eta_1, \eta_2) = 0 \quad \forall \quad \eta_1, \eta_2 \in \mathbb{R}^p$ , then  $\xi_2 = \lambda \xi_1$ .
- (2)  $(\mathbb{R}^p, B_{1,p})$  is irreducible.

*Proof.* Let  $g_0$  be the usual Euclidean inner product;  $g_0(X_i, X_j) := \delta_{ij}$ . Then:

$$B_{1,p}(\eta_1, \eta_2, \eta_3, \eta_4) = g_0(\eta_1, \eta_4)g_0(\eta_2, \eta_3) - g_0(\eta_1, \eta_3)g_0(\eta_2, \eta_4).$$

Let  $O(p)$  be the usual Euclidean orthogonal group. If  $\theta \in O(p)$ , then  $\theta^* B_{1,p} = B_{1,p}$ . By applying a suitable element of  $\theta \in O(p)$  and rescaling if necessary, we may assume without loss of generality  $\xi_1 = X_1$  in proving Assertion (1). We expand  $\xi_2 = \sum_{1 \leq i \leq p} a_i X_i$ . For  $i > 1$ ,  $a_i = R(\xi_1, \xi_2, X_i, X_1) = 0$ . Assertion (1) follows.

Suppose that we have a non-trivial decomposition  $\mathbb{R}^p = W_1 \oplus W_2$  which induces a decomposition  $B_{1,p} = B_{1,p}^1 \oplus B_{1,p}^2$ . Let  $0 \neq \xi_i \in W_i$ . Then  $B_{1,p}(\xi_1, \xi_2, \cdot, \cdot) = 0$  so, by Assertion (1),  $\xi_1 = \lambda \xi_2$ ; this is false.  $\square$

*Proof of Theorem 1.3 (4).* We showed in Section 3.1 that  $\mathcal{U}_{1,p}$  is a 0-model for  $\mathcal{M}_{1,p,f}$ . Thus we must only show that  $\mathcal{U}_{1,p}$  is irreducible. Let

$$K := \text{Span}\{Y_1, \dots, Y_p\} = \{\eta \in \mathbb{R}^{2p} : R(\xi_1, \xi_2, \xi_3, \eta) = 0 \quad \forall \quad \xi_i \in \mathbb{R}^{2p}\}.$$

Let  $\pi$  be the natural projection from  $\mathbb{R}^{2p}$  to  $\mathbb{R}^p = \text{Span}\{X_1, \dots, X_p\} = \mathbb{R}^{2p}/K$ . We then have that  $A_{1,p} = \pi^* B_{1,p}$ . Suppose there is a non-trivial decomposition:

eqn-4.a

$$(4.a) \quad \mathbb{R}^{2p} = V_1 \oplus V_2, \quad g_{1,p} = g_{1,p}^1 \oplus g_{1,p}^2, \quad \text{and} \quad A_{1,p} = A_{1,p}^1 \oplus A_{1,p}^2.$$

We argue for a contradiction. Since  $V_1 \perp V_2$ , the metrics  $g_{1,p}^i$  are non-trivial on  $V_i$ . In particular, the subspaces  $V_i$  are not totally isotropic. Equation (4.a) induces a corresponding decomposition

$$\mathbb{R}^p = V_1/\{K \cap V_1\} \oplus V_2/\{K \cap V_2\} \quad \text{and} \quad B_{1,p} = B_{1,p}^1 \oplus B_{1,p}^2.$$

By Assertion (1), this decomposition of  $\mathbb{R}^p$  is trivial; we assume that the notation is chosen so  $V_2/\{K \cap V_2\} = \{0\}$  and hence  $V_2 \subset K$  so  $V_2$  is totally isotropic. This is a contradiction.  $\square$

**4.2. The model  $\mathcal{M}_{2,s,F}$ .** We adopt the notation of Section 1.2. We define an algebraic curvature tensor  $B_{2,s}$  on  $\mathbb{R}^{2s} := \text{Span}\{U_1, \dots, U_s, T_1, \dots, T_s\}$  by setting:

$$B_{2,s}(U_i, U_j, U_k, T_l) = \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}.$$

lem-4.2

**Lemma 4.2.**

- (1) Let  $0 \neq \xi_1, \xi_2 \in \mathbb{R}^{2s}$ . If  $B_{2,s}(\xi_1, \xi_2, \eta_1, \eta_2) = 0$  and  $B_{2,s}(\xi_1, \eta_1, \eta_2, \xi_2) = 0$  for all  $\eta_1, \eta_2 \in \mathbb{R}^{2s}$ , then  $\xi_1, \xi_2 \in \text{Span}\{T_1, \dots, T_s\}$ .
- (2)  $(\mathbb{R}^{2s}, B_{2,s})$  is irreducible.



*Proof.* We extend  $\theta \in O(s)$  to act diagonally on  $\mathbb{R}^{2s} = \mathbb{R}^s \oplus \mathbb{R}^s$ ; we then have  $\theta^* B_{2,s} = B_{2,s}$ . Suppose that  $\xi_1 \notin \text{Span}\{T_1, \dots, T_s\}$ . By applying a suitably chosen element  $\xi \in O(s)$  and rescaling if necessary, we may assume without loss of generality

$$\begin{aligned}\xi_1 &= U_1 + b_1 T_1 + \dots + b_s T_s \quad \text{and} \\ \xi_2 &= c_1 U_1 + \dots + c_s U_s + d_1 T_1 + \dots + d_s T_s\end{aligned}$$

for suitably chosen constants  $\{b_1, \dots, b_s, c_1, \dots, c_s, d_1, \dots, d_s\}$ . Let  $i > 1$ . We have

$$0 = B_{2,s}(\xi_1, \xi_2, U_i, T_1) = c_i \quad \text{and} \quad 0 = B_{2,s}(\xi_1, U_i, T_i, \xi_2) = c_1.$$

This shows that  $c_1 = 0$  and  $c_i = 0$  so  $\xi_2 = d_1 T_1 + \dots + d_s T_s$ . Furthermore,

$$0 = B_{2,s}(\xi_1, \xi_2, U_i, U_1) = d_i \quad \text{and} \quad 0 = B_{2,s}(\xi_1, U_i, U_i, \xi_2) = d_1.$$

This implies  $\xi_2 = 0$  which is a contradiction. Thus  $\xi_1 \in \text{Span}\{T_1, \dots, T_s\}$ . As the roles of  $\xi_1$  and  $\xi_2$  are symmetric, we conclude  $\xi_2 \in \text{Span}\{T_1, \dots, T_s\}$  as well; Assertion (1) follows.

Suppose given a non-trivial decomposition  $\mathbb{R}^{2s} = W_1 \oplus W_2$  which induces a decomposition  $B_{2,s} = B_{2,s}^1 \oplus B_{2,s}^2$ . Choose  $0 \neq \xi_i \in W_i$ . By Assertion (2),  $\xi_1, \xi_2$  belong to  $\text{Span}\{T_i\}$ . Thus  $W_1 \subset \text{Span}\{T_i\}$  and  $W_2 \subset \text{Span}\{T_i\}$ . Thus  $\mathbb{R}^{2s}$  is contained in  $\text{Span}\{T_i\}$  which is false.  $\square$

*Proof of Theorem 1.4 (4).* We showed in Section 3.2 that  $\mathcal{U}_{2,s}$  is a 0-model for  $\mathcal{M}_{2,s,F}$ . Thus it suffices to show that  $\mathcal{U}_{2,s}$  is irreducible. Let

$$L := \text{Span}\{V_1, \dots, V_s\} = \{\eta \in \mathbb{R}^{3s} : A_{2,s}(\xi_1, \xi_2, \xi_3, \eta) = 0 \quad \forall \quad \xi_i \in \mathbb{R}^{3s}\}.$$

Let  $\pi$  be the natural projection from  $\mathbb{R}^{3s}$  to

$$\mathbb{R}^{2s} := \text{Span}\{U_1, \dots, U_p, T_1, \dots, T_p\} = \mathbb{R}^{3s}/L.$$

We have  $A_{2,s} = \pi^* B_{2,s}$ . Suppose we have a non-trivial decomposition

$$\boxed{\text{eqn-4. b}} \quad (4.b) \quad \mathbb{R}^{3s} = V_1 \oplus V_2, \quad g_{2,s} = g_{2,s}^1 \oplus g_{2,s}^2, \quad \text{and} \quad A_{2,s} = A_{2,s}^1 \oplus A_{2,s}^2.$$

We argue for a contradiction. We argue as above to see  $V_1$  and  $V_2$  are not totally isotropic. Equation (4.b) induces a corresponding decomposition

$$\mathbb{R}^{2s} = V_1/\{L \cap V_1\} \oplus V_2/\{L \cap V_2\} \quad \text{and} \quad B_{2,s} = B_{2,s}^1 \oplus B_{2,s}^2.$$

By Lemma 4.2, this decomposition must be trivial. We assume the notation chosen so that  $V_2 \subset L$ . Thus  $V_2$  is totally isotropic. This is a contradiction.  $\square$

**4.3. The model  $\mathcal{U}_{3,r}$ .** Adopt the notation of Section 1.3. Let  $1 \leq i < r$ . The non-zero entries in the curvature operator are, up to the usual  $\mathbb{Z}_2$  symmetries,

$$\boxed{\text{eqn-4. c}} \quad (4.c) \quad \begin{aligned} A_{3,r}(X, U_r)U_r &= Y, & A_{3,r}(X, U_r)X &= -V_r, \\ A_{3,r}(X, U_i)V_{i+1} &= Y, & A_{3,r}(X, U_i)X &= -U_{i+1}, \\ A_{3,r}(X, V_{i+1})U_i &= Y, & A_{3,r}(X, V_{i+1})X &= -V_i. \end{aligned}$$

If  $\xi \in \mathbb{R}^{2r+2}$ , then:

$$\begin{aligned} J(\xi)X &\in \text{Span}\{U_2, \dots, U_r, V_1, \dots, V_r, Y\}, \\ J(\xi)U_r &\in \text{Span}\{V_r, Y\}, & J(\xi)U_i &\in \text{Span}\{U_{i+1}, Y\}, \\ J(\xi)V_i &\in \text{Span}\{V_{i-1}, Y\}, & J(\xi)Y &= J(\xi)V_1 = 0. \end{aligned}$$

*Proof of Theorem 1.6 (3).* Display (4.c) shows  $J(\xi)^{2r} = 0$ . As  $J(X)^{2r-1}U_1 = V_1$ ,  $\mathcal{U}_{3,r}$  is  $2r$ -Osseman nilpotent; Theorem 1.6 (3) follows.  $\square$

*Proof of Theorem 1.5 (4).* We showed in Section 3.3 that  $\mathcal{U}_{3,r}$  is a 0-model for  $\mathcal{M}_{3,r,\psi}$ . Thus it suffices to show  $\mathcal{U}_{3,r}$  is irreducible. We suppose the contrary and argue for a contradiction. Suppose there is a non-trivial decomposition

$$\text{eqn-4.d} \quad (4.d) \quad \mathbb{R}^{2r+2} = W_1 \oplus W_2, \quad g_{3,r} = g_{3,r}^1 \oplus g_{3,r}^2, \quad \text{and} \quad A_{3,r} = A_{3,r}^1 \oplus A_{3,r}^2.$$

As above, neither  $W_1$  nor  $W_2$  can be totally isotropic. Decompose  $X = X_1 + X_2$ . Then either  $J(X_1)$  or  $J(X_2)$  is nilpotent of order  $2r$ ; we may assume without loss of generality that the notation is chosen so that  $J(X_1)$  is nilpotent of order  $2r$ . Since  $J(X_1)X_1 = 0$ , this implies  $\dim(W_1) \geq 2r + 1$ . Since the decomposition is assumed non-trivial, this implies  $\dim(W_2) = 1$ . Let  $\xi$  span  $W_2$ ;  $\xi$  can not be a null vector since  $W_2$  is not totally isotropic. On the other hand since  $\dim(W_2) = 1$ ,  $A_{3,r}(\eta_1, \eta_2)\xi = 0$  for  $\eta_i \in W_2$ . The decomposition of Equation (4.d) then shows  $A_{3,r}(\eta_1, \eta_2)\xi = 0$  for all  $\eta_1, \eta_2 \in \mathbb{R}^{2r+2}$ . This implies  $\xi \in \text{Span}\{V_1, Y\}$  which is a totally isotropic subspace; this is a contradiction.  $\square$

## 5. HOMOGENEITY

sect-5

**5.1. The manifolds  $\mathcal{M}_{1,p,f}$ .** If  $\phi$  is a symmetric bilinear form on  $V$ , then we may define an algebraic curvature tensor  $R(\phi)$  on  $V$  by setting:

$$\text{eqn-5.a} \quad (5.a) \quad R(\phi)(\xi_1, \xi_2, \xi_3, \xi_4) := \phi(\xi_1, \xi_4)\phi(\xi_2, \xi_3) - \phi(\xi_1, \xi_3)\phi(\xi_2, \xi_4).$$

One then has, see for example the discussion in [4],

lem-5.1

**Lemma 5.1.** *Let  $\phi_1$  and  $\phi_2$  be symmetric positive definite bilinear forms on a vector space  $V$  of dimension at least 3. If  $R(\phi_1) = R(\phi_2)$ , then  $\phi_1 = \phi_2$ .*

*Proof of Theorem 1.3 (6).* Adopt the notation of Section 1.2. Fix  $P \in \mathbb{R}^{2p}$  and let  $V_P := T_P \mathbb{R}^{2p}$ . We consider a 1-model

$$\mathcal{V}_P^1 := (V_P, g_{1,p,f}|_{V_P}, R_{1,p,f}|_{V_P}, \nabla R_{1,p,f}|_{V_P}).$$

Also consider the subspace

$$Y_P := \{\eta \in V_P : R_{1,p,f}(\xi_1, \xi_2, \xi_3, \eta) = 0 \quad \forall \quad \xi_i \in V_P\} = \text{Span}\{\partial_1^y, \dots, \partial_p^y\}.$$

Let  $\pi$  be the natural projection from  $V_P$  to  $X_P := V_P/Y_P$ . As

$$H(\xi_1, \xi_2) = 0, \quad R_{1,p,f}(\xi_1, \xi_2, \xi_3, \xi_4) = 0, \quad \text{and} \quad \nabla R_{1,p,f}(\xi_1, \xi_2, \xi_3, \xi_4; \xi_5) = 0$$

if any  $\xi_i \in Y_P$ , there are structures  $H_{X,P}$ ,  $A_{X,P}$ , and  $A_{X,P}^1$  on  $X_P$  which are characterized by the identities:

$$\pi^* H_{X,P} = H_f|_{V_P}, \quad \pi^* A_{X,P} = R_{1,p,f}|_{V_P}, \quad \text{and} \quad \pi^* A_{X,P}^1 = \nabla R_{1,p,f}|_{V_P}.$$

Assume  $\mathcal{M}_{1,p,f}$  is 1-curvature homogeneous. Let  $P, Q \in \mathbb{R}^{2p}$ . Let  $\Theta$  be an isomorphism from  $\mathcal{V}_P^1$  to  $\mathcal{V}_Q^1$ . It is immediate from the defining relation that  $\Theta(Y_P) \subset Y_Q$ ; a dimension count then implies  $\Theta(Y_P) = Y_Q$ . Consequently  $\Theta$  induces a map  $\tilde{\Theta} : X_P \rightarrow X_Q$  so

$$\tilde{\Theta}^* A_{X,Q} = A_{X,P} \quad \text{and} \quad \tilde{\Theta}^* A_{X,Q}^1 = A_{X,P}^1.$$

We adopt the notation of Equation (5.a) and let  $R(\phi)$  be the curvature tensor defined by a bilinear form  $\phi$ . Since

$$R(H_{X,P}) = A_{X,P} = \tilde{\Theta}^*(A_{X,Q}) = R(\tilde{\Theta}^* H_{X,Q}),$$

Lemma 5.1 implies that  $H_{X,P} = \tilde{\Theta}^* H_{X,Q}$ . Let  $\|\cdot\|_\phi^2$  denote the norm taken with respect to a positive definite bilinear form  $\phi$ . We then have

$$\alpha_1(P) = \|A_{X,P}^1\|_{H_{X,P}}^2 = \|A_{X,Q}^1\|_{H_{X,Q}}^2 = \alpha_1(Q).$$

Consequently  $\alpha_1$  is constant.  $\square$

**5.2. The manifolds  $\mathcal{M}_{2,s,F}$ .** We begin by studying the Lie group associated to the model  $\mathcal{U}_{2,s}$ . We say that  $\mathcal{B} = \{u_1, \dots, u_s, t_1, \dots, t_s, v_1, \dots, v_s\}$  is a *normalized basis* for  $\mathbb{R}^{3s}$  if the normalizations of Equation (1.a) hold, i.e.

$$g_{2,s}(u_i, v_j) = \delta_{ij}, \quad g_{2,s}(t_i, t_j) = -\delta_{ij}, \quad \text{and} \\ A_{2,s}(u_i, u_j, u_k, t_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}.$$

Let  $O(s) \subset M_s(\mathbb{R})$  be the usual orthogonal group of  $s \times s$  matrices;  $\kappa_{ij} \in O(s)$  if and only if  $\sum_k \kappa_{ik}\kappa_{jk} = \delta_{ij}$ .

**lem-5.2**

**Lemma 5.2.** *Let  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  be two normalized bases for  $\mathbb{R}^{3s}$ . If  $s \geq 3$ , then there exists a matrix  $\kappa_1 \in O(s)$  and matrices  $\kappa_2, \kappa_3, \kappa_5 \in M_s(\mathbb{R})$  so that:*

$$\tilde{u}_i = \sum_j \{\kappa_{1,ij}u_j + \kappa_{2,ij}t_j + \kappa_{3,ij}v_j\}, \\ \tilde{t}_i = \sum_j \{\kappa_{1,ij}t_j + \kappa_{5,ij}v_j\}, \quad \text{and} \quad \tilde{v}_i = \sum_j \kappa_{1,ij}v_j.$$

*Proof.* We note that

$$Y : = \{\eta \in \mathbb{R}^{3s} : R_{2,s}(\zeta_1, \zeta_2, \zeta_3, \eta) = 0 \text{ for all } \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}^3\} \\ = \text{Span}\{v_1, \dots, v_s\} = \text{Span}\{\tilde{v}_1, \dots, \tilde{v}_s\}, \quad \text{and} \\ Y^\perp : = \{\eta \in \mathbb{R}^{3s} : g_{2,s}(\eta, \zeta) = 0 \text{ for all } \zeta \in Y\} \\ = \text{Span}\{t_1, \dots, t_s, v_1, \dots, v_s\} = \text{Span}\{\tilde{t}_1, \dots, \tilde{t}_s, \tilde{v}_1, \dots, \tilde{v}_s\},$$

Consequently we may express

$$\tilde{u}_i = \sum_j \{\kappa_{1,ij}u_j + \kappa_{2,ij}t_j + \kappa_{3,ij}v_j\}, \\ \tilde{t}_i = \sum_j \{\kappa_{4,ij}t_j + \kappa_{5,ij}v_j\}, \quad \text{and} \quad \tilde{v}_i = \sum_j \kappa_{6,ij}v_j.$$

We verify that  $\kappa_4 \in O(s)$  by checking

$$-\delta_{ij} = g_{2,s}(\tilde{t}_i, \tilde{t}_j) = \sum_{k,l} \kappa_{4,ik}\kappa_{4,jl}g_{2,s}(t_k, t_l) = -\sum_k \kappa_{4,ik}\kappa_{4,jk}.$$

The orthogonal group acts diagonally on  $\mathbb{R}^{3s}$  by

$$\kappa : u_i \rightarrow \sum_j \kappa_{ij}u_j, \quad \kappa : t_i \rightarrow \sum_j \kappa_{ij}t_j, \quad \text{and} \quad \kappa : v_i \rightarrow \sum_j \kappa_{ij}v_j.$$

This action preserves the structures involved. By making a suitable change of basis, therefore, we may suppose without loss of generality that  $\kappa_4 = \text{id}$  in the proof of the Lemma, i.e. that we have:

$$\tilde{u}_i = \sum_j \{\kappa_{1,ij}u_j + \kappa_{2,ij}t_j + \kappa_{3,ij}v_j\}, \\ \tilde{t}_i = t_i + \sum_j \kappa_{5,ij}v_j, \quad \text{and} \quad \tilde{v}_i = \sum_j \kappa_{6,ij}v_j.$$

To show  $\kappa_1 = \text{id}$ , fix  $i \neq j$ . Since  $s \geq 3$ , we can choose

$$0 \neq u \in \text{Span}_{k \neq j} \{u_k\} \cap \text{Span}_{k \neq j; 1 \leq \ell \leq s} \{\tilde{u}_k, \tilde{t}_\ell, \tilde{v}_\ell\}.$$

Expand  $u = \sum_{k \neq j} \varepsilon_k u_k$ . As  $\tilde{t}_j = t_j + \sum_k \kappa_{5,jk}v_k$ , we may compute

$$0 = R_{2,s}(\tilde{u}_i, u, u, \tilde{t}_j) = R_{2,s}(\sum_k \kappa_{1,ik}u_k, \sum_{a \neq j} \varepsilon_a u_a, \sum_{b \neq j} \varepsilon_b u_b, t_j) \\ = \kappa_{1,ij} \sum_{a \neq j} \varepsilon_a^2.$$

This shows  $\kappa_{1,ik} = 0$  for  $i \neq k$  so  $\kappa_1$  is diagonal. Since

$$1 = R_{2,s}(\tilde{u}_i, \tilde{u}_j, \tilde{u}_j, \tilde{t}_i) = \kappa_{1,ii}\kappa_{1,jj}\kappa_{1,jj},$$

and similarly  $1 = \kappa_{1,jj}\kappa_{1,ii}\kappa_{1,ii}$ , we have  $\kappa_{1,ii} = 1$  as desired. The identity  $g_{2,s}(\tilde{u}_i, \tilde{v}_j) = \delta_{ij}$  then shows  $\kappa_6 = \text{id}$  in this special situation.  $\square$

Fix  $P \in \mathbb{R}^{3s}$  and let  $V_P := T_P \mathbb{R}^{3s}$ . Consider a 1-model

$$\mathcal{V}_P^1 := (V_P, g_{2,s,F}|_{V_P}, R_{2,s,F}|_{V_P}, \nabla R_{2,s,F}|_{V_P}).$$

Also consider the subspaces

$$Y_P := \{\eta \in V_P : R_{2,s,F}(\xi_1, \xi_2, \xi_3, \eta) = 0 \ \forall \ \xi_i \in V_P\} = \text{Span}\{\partial_1^v, \dots, \partial_s^v\}, \text{ and}$$

$$Y_P^\perp = \{\eta \in \mathbb{R}^{3s} : g_{2,s,F}(\eta, \xi_1) = 0 \ \forall \ \xi_1 \in Y_P\} = \text{Span}\{\partial_1^t, \dots, \partial_s^t, \partial_1^v, \dots, \partial_s^v\}.$$

Let  $\pi$  be the natural projection from  $V_P$  to  $X_P := V_P/Y_P$ . There is a natural co-variant derivative algebraic curvature tensor  $A_{X,P}^1$  on  $X_P$  so  $\pi^* A_{X,P}^1 = \nabla R_{2,s,F}|_{V_P}$ ;

$$A_{X,P}^1(\tilde{U}_i, \tilde{U}_j, \tilde{U}_j, \tilde{U}_i; \tilde{U}_i) := (\partial_i^u)^3 f_i + 4u_i.$$

The elements  $\{\tilde{U}_i := \pi \partial_i^u\}$  are a basis for  $X_P$ . Define a non-degenerate bilinear form  $L_P$  on  $X_P$  by requiring that

$$L_P(\tilde{U}_i, \tilde{U}_j) = \delta_{ij}.$$

If  $\Theta$  is an isomorphism from  $\mathcal{V}_P^1$  to  $\mathcal{V}_Q^1$ , then clearly  $\Theta(Y_P) = Y_Q$ . Consequently  $\Theta(Y_P^\perp) = Y_Q^\perp$  so  $\Theta$  induces a map  $\tilde{\Theta}$  from  $X_P$  to  $X_Q$ .

To construct the normalized basis of Equation (3.a) we set:

$$U_i = \partial_i^u + \text{Span}_j\{\partial_j^t, \partial_j^v\}, \ T_i = \partial_i^t + \text{Span}_j\{\partial_j^v\}, \text{ and } V_i = \partial_i^v.$$

We apply Lemma 5.2 to expand

$$\Theta U_i = \sum_j \kappa_{1,ij} U_j + \text{Span}_j\{T_j, V_j\}$$

where  $\kappa_1 \in O(s)$ . Since  $\tilde{U}_i = \pi U_i$ ,  $\tilde{\Theta} \tilde{U}_i = \sum_j \kappa_{1,ij} \tilde{U}_j$ . The following Lemma is now immediate:

**lem-5.3**

**Lemma 5.3.** *If  $\Theta$  is an isomorphism from  $\mathcal{V}_P^1$  to  $\mathcal{V}_Q^1$ , then  $\tilde{\Theta}^* L_Q = L_P$ .*

*Proof of Theorem 1.4 (6).* Assume  $\mathcal{M}_{2,s,F}$  is 1-curvature homogeneous. Let  $P$  and  $Q$  be points of  $\mathbb{R}^{3s}$ . Let  $\Theta$  be an isomorphism from  $\mathcal{V}_P^1$  to  $\mathcal{V}_Q^1$ . Since  $\alpha_2 = |A_{X,P}^1|_{L_P}^2$ , Lemma 5.3 implies  $\alpha_2$  must be constant.  $\square$

**5.3. The manifolds  $\mathcal{M}_{3,r,\psi}$ .** Adopt the notation of Section 1.4. Assume that  $\psi''' > 0$  and that  $\psi''' > 0$ . Set

$$K_P := \{\xi \in \mathbb{R}^{2r+2} : \exists \xi_i \in T_P \mathbb{R}^{2r+2} \text{ so } \nabla^2 R_{3,r,\psi}(\xi_1, \xi_2, \xi_3, \xi_4; \xi_5, \xi) \neq 0\}.$$

*Proof of Theorem 1.5 (6).* Assume that  $\psi'' > 0$  and  $\psi''' > 0$  for all points of  $\mathbb{R}$ . The possibly non-zero entries in  $\nabla^2 R_{3,r,\psi}$  are given by:

$$\begin{aligned} \nabla^2 R_{3,r,\psi}(\partial_x, \partial_{u_r}, \partial_{u_r}, \partial_x; \partial_x, \partial_x) &= u_{r-1} \psi'''(u_r), \\ \nabla^2 R_{3,r,\psi}(\partial_x, \partial_{u_r}, \partial_{u_r}, \partial_x; \partial_{u_r}, \partial_{u_r}) &= \psi''''(u_r). \end{aligned}$$

We expand  $\xi = \xi_0 \partial_x + \xi_1 \partial_1^u + \dots + \xi_r \partial_r^u + \xi_{r+1} \partial_r^v + \dots + \xi_{2r} \partial_1^v + \xi_{2r+1} \partial_y$ . Then

$$K_P = \begin{cases} \{\xi \in \mathbb{R}^{2r+2} : \xi_0^2 + \xi_r^2 \neq 0\} & \text{if } \psi'''' \neq 0 \text{ and } u_{r-1} \neq 0, \\ \{\xi \in \mathbb{R}^{2r+2} : \xi_r \neq 0\} & \text{if } \psi'''' \neq 0 \text{ and } u_{r-1} = 0, \\ \{\xi \in \mathbb{R}^{2r+2} : \xi_0 \neq 0\} & \text{if } \psi'''' = 0 \text{ and } u_{r-1} \neq 0, \\ \{0\} & \text{if } \psi'''' = 0 \text{ and } u_{r-1} = 0. \end{cases}$$

Suppose that  $\mathcal{M}_{3,r,\psi}$  is curvature 2-homogeneous. Then  $K_P$  is diffeomorphic to  $K_Q$  for any two points  $P$  and  $Q$  of  $\mathbb{R}^{3s}$ . Let  $P = (0, \dots, 1, u_r, 0, \dots, 0)$  and  $Q = (0, \dots, 0, u_r, 0, \dots, 0)$ . Suppose  $\psi''''(u_r) \neq 0$ . Then  $K_P$  is connected and  $K_Q$  is not connected; this is a contradiction. Suppose  $\psi''''(u_r) = 0$ . Then  $K_P$  is non-empty and  $K_Q$  is empty; again, this is a contradiction.  $\square$

## sect-6

## 6. SYMMETRIC SPACES

*Proof of Theorem 1.8.* We extend an argument of E. Cartan's from the Riemannian to the pseudo-Riemannian setting. Let  $\{e_i\}$  be a parallel frame field along a geodesic  $\sigma$ . Then

$$\partial_t R_{ijkl}(t) = \nabla R(e_i, e_j, e_k, e_l; \dot{\sigma}) = 0.$$

Thus  $R(e_i, \dot{\sigma})\dot{\sigma} = c_{ij}e_j$  for suitably chosen constants  $c_{ij}$ . Let  $Y(t)$  be a Jacobi vector field. Express  $Y(t) = a_i(t)e_i(t)$ . Then:

$$\begin{aligned} 0 &= \ddot{Y}(t) + R(Y, \dot{\sigma})\dot{\sigma} = \{\ddot{a}_j(t) + \sum_j a_i(t)c_{ij}\}e_j(t) \quad \text{so} \\ 0 &= \ddot{a}_j(t) + a_i(t)c_{ij} \quad \text{for } 1 \leq j \leq m. \end{aligned}$$

Since  $-a_j(\xi; -t)$  still satisfies the Jacobi equation with the same initial condition,  $a_j(\xi; t) = -a_j(\xi; -t)$  so  $a_j$  is an odd function of  $t$ . Let  $g_{ij} := g(e_i, e_j)$  be independent of  $t$ . Then

$$g(Y_\xi(t), Y_\eta(t)) = g_{ij}a_i(\xi; t)a_j(\eta; t) = g(Y_\xi(-t), Y_\eta(-t))$$

is an even function of  $t$ . Since the geodesic involution takes  $Y_\xi(t)$  to  $-Y_\xi(-t)$ , this shows the geodesic involution is an isometry and establishes the first assertion.

Let  $P, Q \in M$ . We suppose  $P \neq Q$ . Since  $\exp_P$  is a diffeomorphism from  $T_P M$  to  $M$ , we can choose a geodesic  $\sigma$  so  $\sigma(0) = P$  and  $\sigma(1) = Q$ . Let  $R = \sigma(\frac{1}{2})$ . Then the geodesic involution centered at  $R$  interchanges  $P$  and  $Q$  and is an isometry. Thus  $M$  is a homogeneous space.  $\square$

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